

COMPUTING TROPICAL POINTS AND TROPICAL LINKS

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ABSTRACT. In this article, we construct fast algorithms for computing non-trivial points on and codimension one links of tropical varieties based on triangular decomposition and Newton polygons. Using them, we show that the tropical Grassmannians $\mathcal{G}_{3,8}$ and $\mathcal{G}_{3,9}$ are not simplicial and have links of higher valency.

1. INTRODUCTION

Given an affine variety X over an algebraically closed field K with non-trivial valuation, its tropical variety $\text{Trop}(X)$ is the euclidean closure of its image under component-wise valuation. Explicit computation of tropical varieties is not only of interest for practical applications, but also of theoretical importance on many occasions [SS04; HJJS09; BJMS15; CM16]. However, computing tropical varieties is an algorithmically challenging task, requiring sophisticated techniques from computer algebra and convex geometry, as shown by Bogart, Jensen, Speyer, Sturmfels and Thomas [BJSST07]. In this article, we touch upon two problems that arise in the computation.

The first problem is to pinpoint a *tropical starting point*, a first point on the tropical variety from which all further computations kick off. Up till now, the standard procedure is to traverse the Gröbner complex randomly, checking all vertices for containment in the tropical variety along the way. However, this can be a rather inefficient approach, as there are usually significantly more Gröbner polytopes outside the tropical variety than in the tropical variety.

Another problem that arises repeatedly is to compute *tropical links*, special tropical varieties of simpler combinatorial structure which describe the original tropical variety locally. Their combinatorial structure allows them to be computed by studying the tropical prevariety of any generating set, and successively adding new elements to it until it becomes a tropical basis. While this is a very efficient method for a

Date: December 2016.

2010 *Mathematics Subject Classification.* 14T05, 52B20, 12J25, 13P15.

Key words and phrases. Tropical geometry; Tropical variety; Newton polygon; Computer algebra; Tropical Grassmannian.

The second author was partially supported by the DFG priority project SPP1489 “Algorithmic and Experimental Methods in Algebra, Geometry and Number Theory.

wide range of examples, experiments show that with increasing complexity of the generating set the tropical prevariety computations become infeasible fast.

We present a simple yet novel approach for solving the aforementioned problems, based on the following bread-and-butter techniques in computer algebra and number theory:

- (1) intersecting with hyperplanes to reduce to dimension zero,
- (2) applying triangular decomposition to reduce to triangular sets,
- (3) reading off valuations of roots from Newton polygons.

Moreover, the algorithm for tropical links also relies on recent results by Osserman and Payne on the intersection of tropical varieties [OP13].

We use our algorithms to study some higher tropical Grassmannians. Tropical Grassmannians $\mathcal{G}_{k,n}$ have been studied to great detail by Speyer and Sturmfels [SS04], who showed that $\mathcal{G}_{2,n}$ for $n \geq 2$ and $\mathcal{G}_{3,6}$ are simplicial fans, the former using an intriguing connection to spaces of phylogenetic trees and the latter through explicit computation. In their work on the parametrization and realizability of tropical planes [HJJS09], Hermann, Joswig and Speyer showed that $\mathcal{G}_{3,7}$ is also a simplicial fan. We will complement these findings by showing that $\mathcal{G}_{3,8}$ and $\mathcal{G}_{4,8}$ have both maximal cones which are not simplicial as well as codimension one links which are of higher valency.

All algorithms have been implemented in the computer algebra system SINGULAR [DGPS16], as part of the library `tropicalNewton.lib` [HR16], which will be publicly available with the next SINGULAR release. The convex computations are done using interfaces to GFANLIB [Jen11; JRS16] and POLYMAKE [GJ00; MR16].

We would like to thank Michael Joswig and Benjamin Schröter for their helpful comments.

Convention 1.1

For the remainder of the article, let K be a field with non-trivial valuation $\nu : K \rightarrow \mathbb{R} \cup \{\infty\}$, residue field \mathfrak{K} and uniformizing parameter $p \in \mathcal{O}_K$. We assume K to be algebraically closed, and, for sake of clarity and by abuse of notation, we will also use ν to denote the component-wise valuation $K^n \rightarrow (\mathbb{R} \cup \{\infty\})^n$. We also fix a multivariate polynomial ring $K[x] := K[x_1, \dots, x_n]$.

2. COMPUTING ZERO-DIMENSIONAL TROPICAL VARIETIES

In this section we present an algorithm for computing zero-dimensional tropical varieties using triangular decomposition and Newton polygons. For sake of simplicity, we restrict ourselves to the task of computing a single point on the tropical variety, as the structure of the algorithm easily suggests how the entire tropical variety can be computed with proper bookkeeping. We conclude the section by showing that any generic triangular set admits what we call unique series of Newton polygons, which is the best case for our algorithm.

Definition 2.1

Let $w \in \mathbb{R}^n$ be a weight vector. Given a polynomial $f \in K[x]$, say $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha \cdot x^\alpha$, we define its *initial form* with respect to w to be

$$\text{in}_w(f) = \sum_{w \cdot \alpha + \nu(c_\alpha) \text{ minimal}} \overline{c_\alpha \cdot p^{-\nu(c_\alpha)}} \cdot x^\alpha \in \mathfrak{K}[x].$$

Additionally, we define the evaluation of the tropical polynomial of f at w to be the minimum occurring weighted valued degree:

$$\text{trop}(f)(w) := \min\{w \cdot \alpha + \nu(c_\alpha) \mid c_\alpha \neq 0\}.$$

Given an ideal $I \subseteq K[x]$, we define its *initial ideal* with respect to w to be

$$\text{in}_w(I) = \langle \text{in}_w(f) \mid f \in I \rangle \subseteq \mathfrak{K}[x].$$

The *tropical variety* of I is given by

$$\text{Trop}(I) := \{w \in \mathbb{R}^n \mid \text{in}_w(I) \text{ monomial free}\}.$$

For elements $f \in K[x]$ and finite subsets $F \subseteq K[x]$, we abbreviate $\text{Trop}(f) := \text{Trop}(\langle f \rangle)$ and $\text{Trop}(F) := \text{Trop}(\langle F \rangle)$ respectively.

While the algorithms developed in [BJSST07] work with the aforementioned definition of tropical varieties, the algorithms in this article focus on the following characterization of tropical varieties:

Theorem 2.2 ([MS15, Theorem 3.2.5])

If K is a field with non-trivial valuation ν , then for any $I \subseteq K[x]$ and its corresponding affine variety $X \subseteq K^n$ we have

$$\text{Trop}(I) = \overline{\nu(X \cap (K^*)^n)}$$

where $\overline{(\cdot)}$ denotes the closure in the euclidean topology.

We now describe how to exploit this geometric characterization algorithmically using triangular sets and Newton polygons.

Definition 2.3

A set of polynomials $F = \{f_1, \dots, f_n\} \subseteq K[x]$ is called a *triangular set*, if for each $i = 1, \dots, n$ we have

$$f_i \in K[x_1, \dots, x_i] \setminus K[x_1, \dots, x_{i-1}].$$

Proposition 2.4 ([GP08, Corollary 4.7.4])

Let I be a zero-dimensional ideal, then there exist triangular sets F_1, \dots, F_s such that

- (1) $\sqrt{I} = \bigcap_{i=1}^s \sqrt{\langle F_i \rangle}$,
- (2) $\langle F_i \rangle + \langle F_j \rangle = \langle 1 \rangle$ for $i \neq j$.

Remark 2.5

Triangular decompositions as in Proposition 2.4 were initially introduced by Lazard [Laz92]. While a primary decomposition always yields a triangular decomposition, the latter can generally be obtained easier through other methods, see [Laz92, Procedure 1] or [GP08, Algorithm 4.7.8] for details.

Definition 2.6

For a univariate polynomial $f \in K[x_k]$, say $f = \sum_{i=0}^d c_i x_k^i$ with $c_i \in K$, $c_0 c_d \neq 0$, the *Newton polygon* (or *extended Newton polyhedron*) is defined to be

$$\Delta(f) := \text{Conv}(\{(i, \nu(c_i)) \mid i = 0, \dots, d\} \cup \{(0, \infty)\}).$$

Similarly, for a multivariate polynomial $f \in K[x_1, \dots, x_k]$, say $f = \sum_{i=0}^d f_i \cdot x_k^i$ with $f_i \in K[x_1, \dots, x_{k-1}]$ and a weight $w \in \mathbb{R}^{k-1}$, we define the *expected Newton polygon* of f at w to be

$$\Delta_w(f) := \text{Conv}(\{(i, \text{trop}(f_i)(w)) \mid i = 0, \dots, d\} \cup \{(0, \infty)\}).$$

And we say f has a *unique Newton polygon* at w , if for all vertices $(i, \text{trop}(f_i)(w)) \in \Delta_w(f)$ the initial form $\text{in}_w(f_i)$ is a monomial. We write $\Lambda(f)$ or $\Lambda_w(f)$ for the set of all negative slopes of $\Delta(f)$ or $\Delta_w(f)$ respectively.

Proposition 2.7

For a polynomial $f \in K[x_1, \dots, x_k]$ and weight $w \in \mathbb{R}^{k-1}$ the following are equivalent:

- (1) f has a unique Newton polygon at w ,
- (2) for all $z \in K^{k-1}$ with $\nu(z) = w$ we have $\Delta(f(z, x_k)) = \Delta_w(f)$.

Proof. Note that for any coefficient $c \in K$ and any substitute $z \in K^{k-1}$ with $\nu(z) = w$ and any exponent vector $\alpha \in \mathbb{N}^{k-1}$ we have $\nu(c \cdot z^\alpha) = w \cdot \alpha + \nu(c) = \text{trop}(c \cdot x^\alpha)(w)$. Hence for any $f_i \in K[x_1, \dots, x_{k-1}]$ we always have

$$\nu(f_i(z)) \geq \text{trop}(f_i)(w),$$

with equality guaranteed if $\text{in}_w(f_i)$ is a monomial, i.e. (1) implies (2).

For the converse, it suffices to show that the equality is guaranteed only if f_i is a monomial. Since K is algebraically closed, so is its residue field \mathfrak{K} . In particular, if $\text{in}_w(f_i)$ is no monomial, then it has a non-zero root in \mathfrak{K}^{k-1} . Picking any $z \in K^{k-1}$ with $\nu(z) = w$ and $\text{in}_w(f)(z_1 \cdot p^{-\nu(z_1)}, \dots, z_{k-1} \cdot p^{-\nu(z_{k-1})}) = 0$, $p \in K$ denoting a uniformizing parameter, yields $\nu(f_i(z)) \geq \text{trop}(f_i)(w)$. \square

The usefulness of Newton polygons rests on the following well-known result, which allows us to read off points on tropical varieties from the Newton polygons.

Lemma 2.8 ([Neu99, Proposition II.6.3])

Let f be a univariate polynomial over K and e an edge of the Newton polygon $\Delta(f)$ connecting vertices (r, y) and (s, z) with slope $-m$. Then f has exactly $s - r$ roots with valuation m .

Example 2.9

The polynomial $f = 2^3x_3^2 + (x_1 - x_2)x_3 + (x_1^2 - 2x_2) \in \overline{\mathbb{Q}_2}[x]$ has an exact Newton polygon at all $(w_1, w_2) \in \mathbb{R}^2$ with $w_1 \neq w_2$ and $2w_1 \neq w_2 + 1$.

For example, for all $z_1, z_2 \in \overline{\mathbb{Q}_2}$ with $\nu_2(z_1) = 2$ and $\nu_2(z_2) = 1$, the Newton polygon $\Delta(f(z_1, z_2, x_3))$ will have vertices at $(0, 2)$, $(1, 1)$ and $(2, 3)$. Using Lemma 2.8 we conclude that there exists $z_3 \in \overline{\mathbb{Q}_2}$ with $v(z_3) = 1$ such that $f(z_1, z_2, z_3) = 0$, that is $(2, 1, 1) \in \text{Trop}(f)$.

On the other hand, for $z_1, z_2 \in \overline{\mathbb{Q}_2}$ with $\nu_2(z_1) = 0$ and $\nu_2(z_2) = 0$, the Newton polygon $\Delta(f(z_1, z_2, x_3))$ may vary depending on the choice of z_1, z_2 , as illustrated in Figure 1.

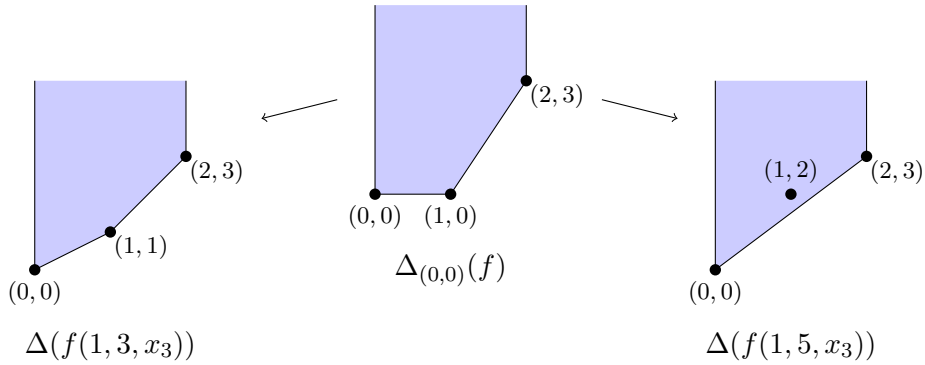


FIGURE 1. expected and possible Newton polygons of f .

Algorithm 2.10 (zero-dimensional tropical varieties)

Input: $F = \{f_1, \dots, f_n\} \subseteq K[x]$ a triangular set such that its affine variety $V(F)$ is zero-dimensional and contained in $(K^*)^n$

Output: $w \in \text{Trop}(F)$

- 1: Pick $w_1 \in \Lambda(f_1)$.
- 2: **for** $i = 2, \dots, n$ **do**
- 3: **if** f_i has a unique Newton polygon at (w_1, \dots, w_{i-1}) **then**
- 4: Pick $w_i \in \Lambda_{(w_1, \dots, w_{i-1})}(f_i)$.
- 5: **else**
- 6: Compute a root $(c_1, \dots, c_{i-1}) \in V(f_1, \dots, f_{i-1})$
- 7: Pick $w_i \in \Lambda(f_i(c_1, \dots, c_{i-1}, x_i))$.
- 8: **return** (w_1, \dots, w_n)

Proof. The termination of the algorithm is clear and the correctness follows directly from Proposition 2.7 and Lemma 2.8. However it needs to be shown that all occurring $\Lambda_{(w_1, \dots, w_{i-1})}(f_i)$ are non-empty, i.e. that the respective Newton polygons have non-zero width.

This follows from the conditions on $V(F)$: Suppose $f_i = h_i \cdot x_i^d$ for some $h_i \in K[x_1, \dots, x_{i-1}] \subseteq K[x_1, \dots, x_n]$ and some $d \in \mathbb{N}$. If $h_i(z) = 0$ for some $z \in$

$V(f_1, \dots, f_{i-1})$, then $z + \sum_{j=i}^n K \cdot e_j \subseteq V(f_1, \dots, f_{i-1})$ contradicting $\dim(V(F)) = 0$. Hence $h_i(z) \neq 0$ for all $z \in V(f_1, \dots, f_{i-1})$, implying that $z_i = 0$ for all $z \in V(f_1, \dots, f_{i-1})$ and contradicting $V(F) \subseteq (K^*)^n$. \square

Remark 2.11

Most of Algorithm 2.10 is straight forward, but performing Step 6 is a rather delicate task and depends very much on the field over which the polynomials are defined. For example, if the polynomials in F are given over a finite extension of a p -adic field \mathbb{Q}_p , one can use Hensel Lifting to find its roots, however it might be necessary to pass to its splitting fields (see [Yok97; GK00]) to ensure their existence. For all examples in this articles, this was unnecessary since either the Newton polygons were unique or the equations were of a form from which the roots could be easily read off.

Example 2.12 (computing of roots with finite precission)

Note that at times it suffices to compute the root in Step 6 of Algorithm 2.10 with finite precision. Consider the triangular set $F = \{f_1, f_2, f_3\} \subseteq \overline{\mathbb{Q}_3}[x_1, x_2, x_3]$, where

$$f_1 = x_1^2 + 3x_1 - 1, \quad f_2 = x_2^2 + 9x_2 - 1, \quad f_3 = 3x_3^2 + (x_1 - x_2)x_3 + 1.$$

From the Newton polygons of f_1 and f_2 we see that elements $z_1, z_2 \in \overline{\mathbb{Q}_3}$ with $f_1(z_1) = f_2(z_2) = 0$ must satisfy $\nu_3(z_1) = \nu_3(z_2) = 0$. However, f_3 does not have a unique Newton polygon at $(0, 0)$ and $\Delta(f_3(z_1, z_2, x_3))$ depends on the actual values of z_1 and z_2 . More precisely, we have

$$\Delta(f_3(z_1, z_2, x_3)) = \begin{cases} \begin{array}{c} \text{Quadrilateral with vertices } (0,0), (1,0), (2,1), \text{ and } (0,1) \\ \text{shaded blue} \end{array} & \text{if } \nu_3(z_1 - z_2) = 0, \\ \begin{array}{c} \text{Triangle with vertices } (0,0), (2,1), \text{ and } (0,1) \\ \text{shaded blue} \end{array} & \text{if } \nu_3(z_1 - z_2) > 0. \end{cases}$$

Applying Hensel Lifting we see that f_1 has a root $z_1 \in \mathbb{Z}_3$ with $z_1 \equiv 4 \pmod{3^2\mathbb{Z}_3}$ and f_2 has a root $z_2 \in \mathbb{Z}_3$ with $z_2 \equiv 1 \pmod{3^2\mathbb{Z}_3}$. Since for these roots we have $z_1 - z_2 \not\equiv 0 \pmod{3\mathbb{Z}_3}$ and $z_1 - z_2 \in 3\mathbb{Z}_3$, we are in the second case and there exists $z_3 \in \overline{\mathbb{Q}_3}$ with $\nu_3(z_3) = -\frac{1}{2}$ and $f_3(z_1, z_2, z_3) = 0$. We conclude that $(0, 0, -\frac{1}{2}) \in \text{Trop}(\langle F \rangle)$.

Example 2.13 (computing entire tropical varieties)

As mentioned in the beginning of the section, Algorithm 2.10 can also be used to compute entire tropical varieties in the zero-dimensional case, if all options are exhausted in Steps 4 and 6 to 7.

Consider in the triangular set $F = \{f_1, f_2, f_3\} \subseteq \mathbb{C}\{\{t\}\}[x_1, x_2, x_3]$, where

$$f_1 = tx_1^2 + x_1 + 1, \quad f_2 = tx_1x_2^2 + x_1x_2 + 1, \quad f_3 = x_1x_2x_3 + 1.$$

Then F admits several choices for negative slopes throughout the algorithm, and every time the choice induces a new unique Newton polygon as illustrated in Figure 2. Keeping track of all of them, allows us to reconstruct its tropical variety:

$$\text{Trop}(F) = \{(0, 0, 0), (0, -1, 1), (-1, 1, 0), (-1, -1, 2)\}.$$

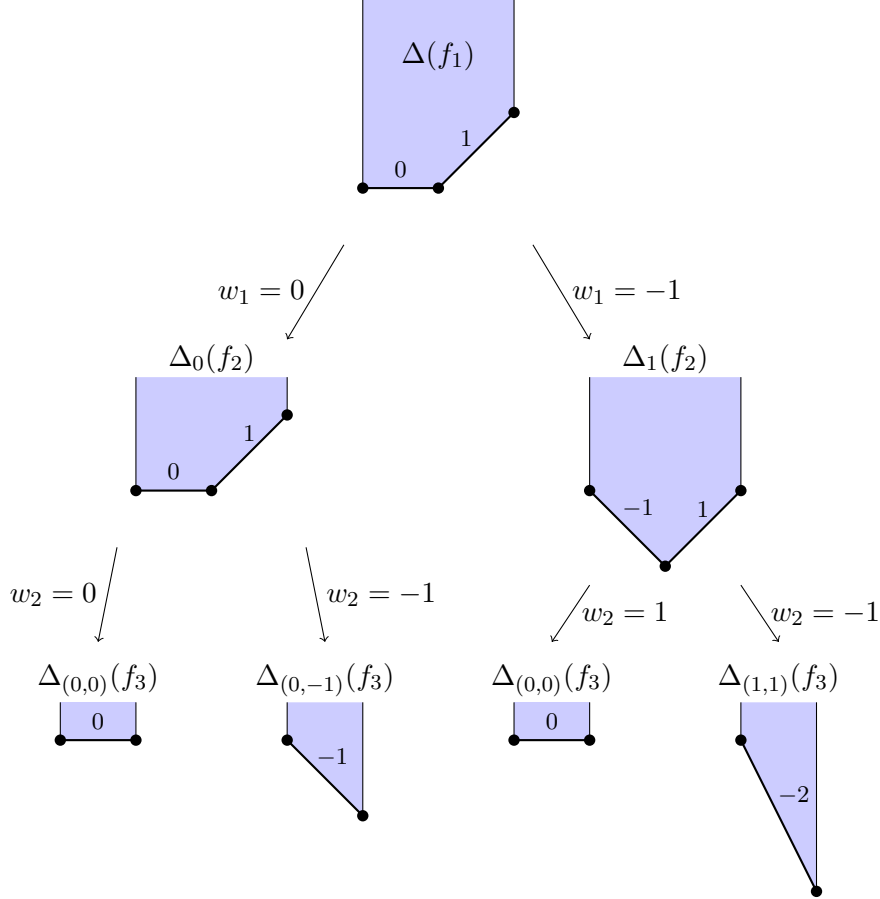


FIGURE 2. possible series of Newton polygons of F .

We conclude this section by showing that any generic triangular set F resembles Example 2.13 in the sense that an approximation of the roots in Step 6 of Algorithm 2.10 will never be necessary regardless of the choices made. This means that Algorithm 2.10 can be used to compute $\text{Trop}(F)$ without having to compute any coordinate of any point of $V(F)$.

Definition 2.14

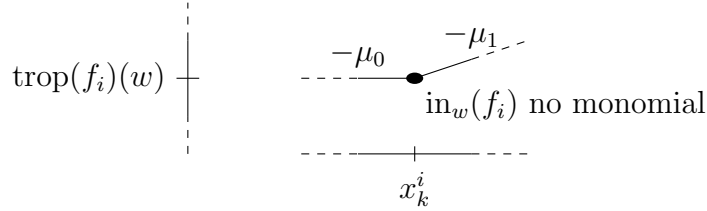
We say a triangular set $F = \{f_1, \dots, f_n\}$ admits *series of unique Newton polygons*, if for all weights $w = (w_1, \dots, w_{k-1}) \in K^{k-1}$ with $w_i \in \Lambda_{(w_1, \dots, w_{i-1})}(f_i)$ for $i = 1, \dots, k-1$ the polynomial f_k has a unique Newton polygon at (w_1, \dots, w_{k-1}) .

Lemma 2.15

Suppose given a weight $w \in \mathbb{R}^{k-1}$ and a polynomial $f \in K[x_1, \dots, x_k] \subseteq K[x_1, \dots, x_n]$ such that $(\{w\} \times \mathbb{R}^{n-k+1}) \cap \text{Trop}(f)$ has codimension k . Then f has a unique Newton polygon at w and

$$(\{w\} \times \mathbb{R}) \cap \text{Trop}(f) = \{w\} \times \bigcup_{w_k \in \Lambda_w(f)} \{w_k\} \times \mathbb{R}^{n-k}.$$

Proof. Without loss of generality, assume that $k = n$. Suppose $f = \sum_{i=0}^d f_i \cdot x_k^i$ with $f_i \in K[x_1, \dots, x_{k-1}]$ and assume that f has no unique Newton polygon at w , i.e., that there exists a vertex $(i, \text{trop}(f_i)(w)) \in \Delta_w(f)$ such that $\text{in}_w(f_i)$ is no monomial. Let μ_0 and μ_1 be the negated slopes of the edges after and before the vertex respectively.



Then, for any $w_k \in (\mu_0, \mu_1)$, we have $\text{in}_{(w, w_k)}(f) = \text{in}_w(f_i) \cdot x_k^i$, which is no monomial. This implies $\{w\} \times (\mu_1, \mu_0) \subseteq \text{Trop}(f)$, contradicting the zero-dimensionality of $\text{Trop}(f)$.

Next, we show the equality. For the “ \supseteq ” inclusion, let μ be a slope of an edge of $\Delta_w(f)$, say connecting the two vertices v_0 and v_1 . Then, writing $e(v_0, v_1)$ for the edge connecting v_0 and v_1 ,

$$\text{in}_{(w, \mu)}(f) = \sum_{(i, \text{trop}(f_i)(w)) \in e(v_0, v_1)} \text{in}_w(f_i) \cdot x_k^i.$$

For the converse inclusion, let $(w, w_k) \in \text{Trop}(f)$. It is clear, that for some bounded proper face $e \leq \Delta_w(f)$,

$$\text{in}_{(w, w_k)}(f) = \sum_{(i, \text{trop}(f_i)(w)) \in e} \text{in}_w(f_i) \cdot x_k^i.$$

Note that e cannot be zero-dimensional, as otherwise $\text{in}_{(w, w_k)}(f) = \text{in}_{(w, w'_k)}(f)$ for all $w'_k \in \mathbb{R}$, contradicting the zero-dimensionality of $\text{Trop}(f)$. Hence, e has to be an edge and, consequently, w_k is the slope of e . \square

Proposition 2.16

For a triangular set $F = \{f_1, \dots, f_n\} \subseteq K[x]$ the following are equivalent:

- (1) $\dim \bigcap_{i=1}^k \text{Trop}(f_i) = n - k$ for all $k = 1, \dots, n$.
- (2) F is a tropical basis,

Moreover, if F is a tropical basis, then it admits series of unique Newton polygons.

Proof. We first show that (1) implies that F is a tropical basis and that it admits series of unique Newton polygons. By definition, $\text{Trop}(f_1) = \bigcup_{w_1 \in \Lambda(f_1)} \{w_1\} \times \mathbb{R}^{n-1}$.

Applying Lemma 2.15 repeatedly, we see that for $w_1 \in \Lambda(f_1)$, the polynomial f_2 has a unique Newton polygon at w_1 with

$$(\{w_1\} \times \mathbb{R}^{n-1}) \cap \text{Trop}(f_2) = \{w_1\} \times \bigcup_{w_2 \in \Lambda_{w_1}(f_2)} \{w_2\} \times \mathbb{R}^{n-2},$$

and, for $w_1 \in \Lambda(f_1)$ and $w_2 \in \Lambda_{w_1}(f_2)$, f_3 has a unique Newton polygon at (w_1, w_2) with

$$(\{(w_1, w_2)\} \times \mathbb{R}^{n-2}) \cap \text{Trop}(f_3) = \{(w_1, w_2)\} \times \bigcup_{w_3 \in \Lambda_{(w_1, w_2)}(f_3)} \{w_3\} \times \mathbb{R}^{n-3},$$

and so forth. This shows on the one hand that F admits series of unique Newton polygons and on the other hand that any point in $\bigcap_{i=1}^n \text{Trop}(f_i)$ corresponds to the component-wise valuation of a point in $V(F)$, implying that F is a tropical basis.

It remains to show that if (1) is not true, then F is no tropical basis. Assume for sake of simplicity that $\dim \text{Trop}(f_1) \cap \text{Trop}(f_2) = n - 1$. Because $\text{Trop}(f_1) = \bigcup_{w_1 \in \Lambda(f_1)} \{w_1\} \times \mathbb{R}^{n-1}$ and $\text{Trop}(f_2)$ is invariant under translation by $\{(0, 0)\} \times \mathbb{R}^{n-2}$, there necessarily exist

$$\{\lambda\} \times [\mu_1, \mu_2] \times \mathbb{R}^{n-2} \subseteq \text{Trop}(f_1) \cap \text{Trop}(f_2),$$

for $\lambda \in \Lambda(f_1)$ and a nontrivial $[\mu_1, \mu_2] \subseteq \mathbb{R}$. Consequently,

$$\{\lambda\} \times [\mu_1, \mu_2] \times \{(0, \dots, 0)\} \subseteq \bigcap_{i=1}^n \text{Trop}(f_i),$$

and since $\bigcap_{i=1}^n \text{Trop}(f_i)$ is not zero-dimensional, F cannot be a tropical basis of the zero-dimensional ideal it generates. \square

As condition (1) in Proposition 2.16 is generically true, we conclude:

Corollary 2.17

Any generic triangular set is a tropical basis and hence admits series of unique Newton polygons.

3. COMPUTING TROPICAL STARTING POINTS

In this section, we use Algorithm 2.10 to compute non-trivial points on higher-dimensional tropical varieties, which we do by reducing the dimension to zero by intersecting with random hyperplanes. Moreover, we will use the algorithm to sample random maximal Gröbner cones on the tropical Grassmannians $\mathcal{G}_{3,7}, \mathcal{G}_{4,7}, \mathcal{G}_{3,8}, \mathcal{G}_{4,8}$ and show that the latter two have maximal cones which are not simplicial.

Definition 3.1

Let $I \subseteq K[x]$ be an ideal. The *homogeneity space* of I is given by

$$C_0(I) := \{w \in \mathbb{R}^n \mid \text{in}_w(I) = I\}.$$

It can be read off any reduced Gröbner basis and is trivially included in $\text{Trop}(I)$, provided $\text{Trop}(I)$ is non-empty. We call a point $p \in \text{Trop}(I)$ *non-trivial*, if $p \notin C_0(I)$.

Proposition 3.2

Let $I \subseteq K[x]$ be an ideal of dimension d with algebraically independent set $\{x_1, \dots, x_d\}$. Suppose that the affine variety $X = V(I)$ satisfies $X \cap (K^*)^n \neq \emptyset$. Then there exists a non-empty, Zariski open subset $U \subseteq (K^*)^d$ such that for all $\lambda \in U$

$$\emptyset \neq X \cap V(\langle x_i - \lambda_i \mid i = 1, \dots, d \rangle) \subseteq (K^*)^n$$

and $\dim(X \cap V(\langle x_i - \lambda_i \mid i = 1, \dots, d \rangle)) = 0$.

Proof. Without loss of generality we may assume that X is irreducible. Abbreviating $H_\lambda := V(\langle x_i - \lambda_i \mid i = 1, \dots, d \rangle)$, it is clear that there exists a Zariski open $U_0 \subseteq (K^*)^d$ with $\emptyset \neq X \cap H_\lambda$ and $\dim(X \cap H_\lambda) = 0$. Now consider the set in which the inclusion does not hold. It naturally decomposes into $n - d$ subsets:

$$\begin{aligned} A &:= \{\lambda \in (K^*)^d \mid X \cap H_\lambda \not\subseteq (K^*)^n\} \\ &= \bigcup_{i=d+1}^n \underbrace{\{\lambda \in (K^*)^d \mid \exists z \in X \cap H_\lambda : z_i = 0\}}_{=: A_i}. \end{aligned}$$

As U can be chosen to be $U_0 \setminus \overline{A}$, where $\overline{(\cdot)}$ denotes the Zariski closure in $(K^*)^d$, it suffices to show that $\overline{A_i} \neq (K^*)^d$. This is easy to see: Because X is irreducible and $X \cap (K^*)^n \neq \emptyset$, we necessarily have $\dim(X \cap V(x_i)) < d$ for all $i = d+1, \dots, n$. In particular, $\dim \pi(X \cap V(x_i)) < d$, where $\pi : K^n \rightarrow K^d$ is the canonical projection onto the first d coordinates. And, by construction, $A_i \subseteq \pi(X \cap V(x_i))$. \square

Algorithm 3.3 (tropical starting point)

Input: $I \subseteq K[x]$

Output: $w \in \text{Trop}(I) \setminus C^0(I)$

- 1: Compute a maximal algebraically independent set, say $\{x_1, \dots, x_d\}$, and the homogeneity space $C_0(I)$.
- 2: **repeat**
- 3: Pick $w \in \mathbb{Q}^d$ random with $\{w\} \times \mathbb{R}^{n-d} \cap C_0(I) = \emptyset$.
- 4: Pick $c \in K^d$ random with $\nu(c) = v$.
- 5: Set $I_c := I|_{x_i=c_i} \subseteq K[x_{d+1}, \dots, x_n]$.
- 6: **until** $\dim(I_c) = 0$ and $V(I_c) \subseteq (K^*)^{n-d}$
- 7: Compute a triangular set $F \subseteq K[x_{d+1}, \dots, x_n]$ with $I_c \subseteq \langle F \rangle$.
- 8: Compute $(w_{d+1}, \dots, w_n) \in \text{Trop}(F)$ using Algorithm 2.10.
- 9: **return** $(w_1, \dots, w_d, w_{d+1}, \dots, w_n)$

Example 3.4 (Grass(2, 5))

We consider the Grassmannian $\text{Grass}(2, 5)$, the family of 2-dimensional subspaces in a 5-dimensional vector space over the field of Puiseux series $K := \mathbb{C}\{\{t\}\}$. It is

the vanishing set of the ideal

$$I = \langle x_2x_9 - x_4x_8 + x_5x_7, x_1x_9 - x_3x_8 + x_5x_6, x_0x_9 - x_3x_7 + x_4x_6, \\ x_0x_8 - x_1x_7 + x_2x_6, x_0x_5 - x_1x_4 + x_2x_3 \rangle \subseteq K[x_0, \dots, x_9].$$

Since I is 7-dimensional with maximal independent set $\{x_3, \dots, x_9\}$, the first step is intersecting it with 7 generic affine coordinate hyperplanes in x_3, \dots, x_9 . Choosing $(c_3, \dots, c_9) = (t, \dots, t)$ yields $I_c = \langle x_2, x_1, x_3 \rangle$, giving us the point $(0, 0, 0) \in V(I_c) \not\subseteq (K^*)^3$, which means that our hyperplanes were not suitable for our purposes.

However, choosing $(c_3, \dots, c_9) := (t, t^5, t^3, t^7, t^8, t^2, t^9)$ yields, after a short Gröbner basis calculation, the triangular set

$$x_1 + t^3 - 1, \quad t^6x_2 + t^7 - 1, \quad t^2x_3 + t^4 - 1.$$

Looking at the respective Newton polygons, we conclude that there exist $z_0, z_1, z_2 \in K^*$ with $\nu(z_0) = 0$, $\nu(z_1) = -6$, $\nu(z_2) = -2$ such that $(z_0, z_1, z_2, t, t^5, t^3, t^7, t^8, t^2, t^9) \in V(I)$. Thus

$$(0, -2, -6, 1, 5, 3, 7, 8, 2, 9) \in \text{Trop}(I).$$

Remark 3.5

It is possible to eliminate the randomness in Algorithm 3.3 during the intersection with random affine hyperplanes, by computing stable intersections with affine hyperplanes instead, which can be done thanks to a recent work of Jensen and Yu [JY16]. However, this would require us to extend K transcendently, once for each hyperplane, which is not feasible with high codimension.

In addition to computing starting points for the tropical traversals, the efficiency of Algorithm 3.3 also allows it to be used to test random points on tropical varieties.

Example 3.6 (Grass(k, n) for $k \in \{3, 4\}$ and $n \in \{7, 8\}$)

Using Algorithm 3.3, we sampled random maximal cones on higher tropical Grassmannians. This was done by computing Gröbner cones around random tropical points, dismissing those of lower dimension and duplicates. We analyzed over 1000 maximal cones on each of the tropical varieties of Grass(3, 7), Grass(3, 8) and Grass(4, 7), as well as over 100 maximal cones on the tropical variety of Grass(4, 8). For sake of clarity, we will denote the tropical variety of Grass(k, n) by $\mathcal{G}_{k,n}$.

All checked Gröbner cones were invariant under tensoring with \mathbb{F}_2 : which is not surprising for $\mathcal{G}_{3,7}$. Even though Speyer and Sturmfels showed that $\mathcal{G}_{3,7}$ depends on the characteristic of the ground field, in fact it is the smallest tropical Grassmannian depicting this behavior [SS04, Theorem 3.7], Herrmann, Jensen, Joswig and Sturmfels showed that, out of the 252 000 maximal cones of $\mathcal{G}_{3,7}$, this is only visible on a single cone, the Fano cone [HJJS09, Theorem 2.1].

Of the 1000 Gröbner cones sampled from each of $\mathcal{G}_{3,7}$, every single one was simplicial, which was expected as $\mathcal{G}_{3,7}$ is known to be simplicial [HJJS09, Theorem 2.1]. All sampled cones from $\mathcal{G}_{4,7}$ also turned out to be simplicial, though it is not known

whether $\mathcal{G}_{4,7}$ itself is. In the 1000 and 100 Gröbner cones sampled from $\mathcal{G}_{3,8}$ and $\mathcal{G}_{4,8}$ respectively, each contained exactly one cone which was not simplicial, see the proof of Theorem 3.7.

Not much is known on $\mathcal{G}_{3,8}$ and $\mathcal{G}_{4,8}$, but there is a complete description of the *Dressian* $\text{Dr}(3, 8)$ by Herrmann, Joswig and Speyer [HJS14, Theorem 31], which is a natural tropical prevariety containing $\mathcal{G}_{3,8}$ that parametrizes all tropical linear spaces. It is known all rays of $\text{Dr}(3, 7)$ and $\text{Dr}(3, 8)$ are also rays of $\mathcal{G}_{3,7}$ and $\mathcal{G}_{3,8}$, and that $\mathcal{G}_{3,7}$ contains rays which are not rays of $\text{Dr}(3, 7)$. Our sampling also revealed that this holds for $\mathcal{G}_{3,8}$. In fact, none of the 126 tested rays of $\mathcal{G}_{3,7}$ were rays of $\text{Dr}_{3,8}$, an concrete example is the ray generated by the following vector:

$$(0, -1, 1, 1, -1, 0, 0, 0, 0, 1, 0, 1, 1, 1, 1, 0, 0, 1, 1, -2, 0, 0, -1, -1, 0, -1, -1, 0, -1, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0) \in \mathbb{R}^{\binom{8}{3}}.$$

This is somewhat in stark contrast to $\mathcal{G}_{3,7}$ and $\text{Dr}(3, 7)$, as out of the 721 rays of the Grassmannian 616 were rays of the Dressian [HJJS09, Theorem 2.2].

As an immediate result, we obtain:

Theorem 3.7

The tropical Grassmannian $\mathcal{G}_{d,n}$ is not simplicial for $d = 3, 4$ and $n = 8$.

Proof. Consider the following two points which lie in $\mathcal{G}_{3,8}$ and $\mathcal{G}_{4,8}$ respectively:

$$\begin{aligned} w_{3,8} &= (2, 10, 7, 10, 2, 2, 2, 10, 7, 10, 9, 6, 9, 12, 12, 12, 9, 6, 9, 12, 5, 2, 5, 5, 2, 5, 5, 2, 5, \\ &\quad 1, 7, 7, 7, 7, 7, 9, 6, 9, 12, 12, 12, 9, 6, 9, 12, 9, 7, 9, 12, 7, 7, 7, 7, 7, 7, 7) \in \mathbb{R}^{\binom{8}{3}}, \\ w_{4,8} &= (2, 1, 8, 1, 8, 14, 21, 15, 21, 11, 14, 11, 21, 16, 21, 5, 12, 6, 12, 4, 5, 4, 12, 7, \\ &\quad 12, 17, 12, 17, 19, 19, 19, 17, 12, 17, 19, 14, 21, 15, 21, 11, 14, 11, 21, 16, \\ &\quad 21, 21, 15, 21, 21, 19, 21, 21, 16, 21, 19, 17, 12, 17, 19, 19, 19, 17, 12, 17, \\ &\quad 19, 19, 19, 19, 19, 19) \in \mathbb{R}^{\binom{8}{4}}. \end{aligned}$$

The corresponding reduced Gröbner basis for $\mathcal{G}_{3,8}$ has 686 elements of degrees ranging from 2 to 6, while the reduced Gröbner basis for $\mathcal{G}_{3,9}$ has 1157 elements of degrees ranging from 2 to 8.

The Gröbner cone around $w_{3,8}$ is of dimension 16, generated by 9 rays and a lineality space of dimension 8, and the Gröbner cone around $w_{4,8}$ is of dimension 17, generated by 10 rays and a lineality space of dimension 8. Hence both cones are maximal-dimensional in their respective tropical varieties and not simplicial. \square

We conclude the section with some timings.

Timings 3.8

Figure 3 compares timings of Algorithm 4.12 in [BJSST07], as implemented in GFAN [Jen11], with timings of Algorithm 3.3, as implemented in the SINGULAR library `tropicalNewton.lib`. As GFAN additionally computes the corresponding reduced

Gröbner basis, we also provide analogous timings in SINGULAR. All computations were aborted after exceeding either 7 CPU days or 20 GB RAM.

The following two classes of examples are considered over $\mathbb{C}\{\{t\}\}$:

- (1) Grassmannians $\text{Grass}(k, n)$ families of k -dimensional subspaces in an n -dimensional vector space,
- (2) determinantal varieties $\text{Det}(k, n, m)$ given by the $k \times k$ minors of an $n \times m$ matrix with $n \cdot m$ variables.

	Gröbner method	Newton method	
		$w \in \text{Trop}(I)$	GB under $>_w$
$\text{Det}(2, 5, 5)$	1	1	1
$\text{Det}(3, 5, 5)$	7	1	1
$\text{Det}(2, 6, 6)$	1	1	1
$\text{Det}(3, 6, 6)$	900	8	1
$\text{Det}(4, 6, 6)$	1100	41	1
$\text{Det}(5, 6, 6)$	110	7	1
$\text{Grass}(3, 7)$	-	1	1
$\text{Grass}(3, 8)$	-	3	1
$\text{Grass}(3, 9)$	-	19	12
$\text{Grass}(4, 7)$	-	1	1
$\text{Grass}(4, 8)$	-	9	3
$\text{Grass}(4, 9)$	-	230	900
$\text{Grass}(5, 8)$	-	3	1

FIGURE 3. Timings in seconds, '-' indicate aborted computations.

4. COMPUTING TROPICAL LINKS

In this section, we use Algorithm 2.10 to compute links of tropical varieties around its facets. This is done in two steps. First we intersect the link with a subspace to reduce it to a one-dimensional polyhedral fan. Afterwards, we intersect the fan with affine hypersurfaces to determine all its rays. We also apply the algorithm to show that $\mathcal{G}_{3,8}$ and $\mathcal{G}_{4,8}$ have codimension one links of higher valency.

Definition 4.1

We call $\text{Trop}(I)$ a *tropical curve*, if it is one-dimensional and we say $\text{Trop}(I)$ is combinatorially a curve, if $\text{Trop}(I)/C_0(I)$ is one-dimensional.

Let $u \in \text{Trop}(I)$ sit in the relative interior of a cell of codimension one. Then $\text{Trop}(\text{in}_u(I))$ is referred to as the *tropical link* of I around u . It is the support of a polyhedral fan as well as combinatorially a curve, which describes $\text{Trop}(I)$ locally around w , see Figure 4. The number of maximal Gröbner cones covering it is referred to as its *valency*.

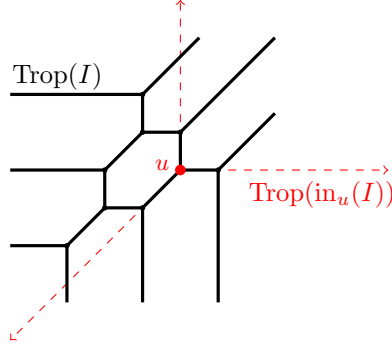


FIGURE 4. a 3-valent link of a tropical cubic curve

The reduction to dimension zero relies on the following result on the intersection of tropical varieties by Osserman and Payne, from which we can directly write down our algorithm.

Theorem 4.2 ([OP13, Theorem 1.1])

Let X and X' be two affine subvarieties. If $\text{Trop}(X) \cap \text{Trop}(X')$ has codimension $\text{codim } \text{Trop}(X) + \text{codim } \text{Trop}(X')$ in a neighborhood of w , then w is contained in $\text{Trop}(X \cap X')$.

Corollary 4.3

Let $\text{Trop}(I)$ be combinatorially a curve with $\dim C_0(I) = d$ and suppose $C_0(I) \cap \text{Lin}(e_{d+1}, \dots, e_n) = \{0\}$. Then for any $c \in (K^)^d$ we have*

$$\text{Trop}(I) \cap \{\nu(c)\} \times \mathbb{R}^{n-d} = \text{Trop}(I + \langle x_i - c_i \mid i = 1, \dots, d \rangle),$$

and $\text{Trop}(I) \cap \{\nu(c)\} \times \mathbb{R}^{n-d}$ is a tropical curve.

Corollary 4.4

Let $\text{Trop}(I)$ be a one-dimensional polyhedral fan. Then for any $c \in K^$ with $\nu(c) \neq 0$ we have*

$$\text{Trop}(I) \cap \{\nu(c)\} \times \mathbb{R}^{n-1} = \text{Trop}(I + \langle x_1 - c \rangle),$$

and $\text{Trop}(I) \cap \{\nu(c)\} \times \mathbb{R}^{n-1}$ is zero-dimensional.

Algorithm 4.5 (tropical links)

Input: $I \trianglelefteq K[x]$ such that $\text{Trop}(I)$ is combinatorially a curve and a polyhedral fan

Output: $W \subseteq \mathbb{R}^n$ such that $\text{Trop}(I) = \bigcup_{w \in W} w \cdot \mathbb{R}_{\geq 0} + C_0(I)$

1: Suppose $\dim(C_0(I)) = d$, assume w.l.o.g.

$$C_0(I) \cap \text{Lin}(e_{d+1}, \dots, e_n) = \{0\}.$$

2: Let J be the image of I under the substitution map

$$K[x_1, \dots, x_n] \rightarrow K[x_d, \dots, x_n], \quad x_i \mapsto \begin{cases} p & \text{if } i < d, \\ x_i & \text{else,} \end{cases}$$

```

3: for  $i = d, \dots, n$  do
4:   Let  $J_i^\pm$  be images of  $J$  under the maps  $x_i \mapsto t^{\pm 1}$  respectively.
5:   Compute  $V_i^\pm = \text{Trop}(J_i^\pm)$  using Algorithm 2.10.
6:   Set

```

$$W_i^\pm := \{(1, \dots, 1, w_d, \dots, w_{i-1}, \pm 1, w_{i+1}, \dots, w_n) \in \mathbb{R}^n \mid (w_d, \dots, w_{i-1}, w_{i+1}, \dots, w_n) \in V_i^\pm\}.$$

```

7:   Scale elements of  $W_i^\pm$  positively so that they are primitive in  $\mathbb{Z}^n$ .
8:   Set  $W := \bigcup_{i=d}^n W_i^\pm$ .
9: return  $W$ 

```

As Algorithm 4.5 relies on Algorithm 2.10, it requires Gröbner bases computations in $n - d$ variables. While Gröbner bases are not known for their graceful degree bounds, the original algorithm [BJSST07, Algorithm 4.8] requires checks whether initial ideals are monomial free, which in turn demand Gröbner bases computations in n variables. Moreover, Joswig and Schröter have shown that the resulting tropical basis satisfies an equally bad degree bound [JS15, Theorem 5]. While it is unclear whether the bound is sharp, it is currently the only bound known. For more information on the complexity of tropical computations, see [The06].

The idea of computing tropical links by reducing the dimension is not new. Based on techniques developed by Hept and Theobald [HT09], Andrew Chan has designed an algorithm which computes tropical links via projection and reconstruction [Cha13]. However, the projections require eliminations of variables, which in turn demand Gröbner bases computations in n variables. Moreover, even the computation of one-dimensional tropical varieties is a highly non-trivial task.

Let $K = \mathbb{C}\{\{t\}\}$ and consider the ideal $I \trianglelefteq K[p_\Gamma \mid \Gamma \subseteq \{1, \dots, 9\}, |\Gamma| = 4]$ generated by the Plücker relations for $\text{Grass}(4, 9)$. Living in an ambient space of dimension 126, its tropical variety is of dimension 21 with a homogeneity space of dimension 9. Using Algorithm 2.10, one possible non-trivial tropical point computed is

[illegible]

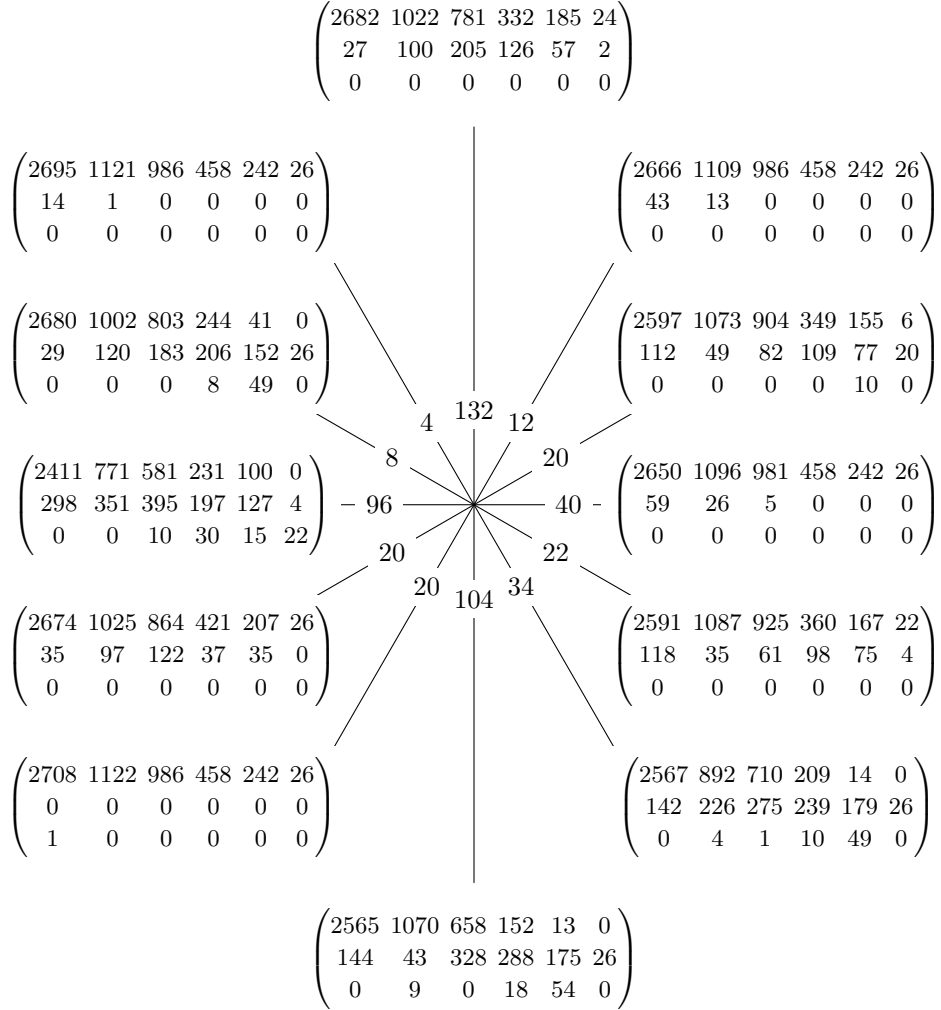


FIGURE 5. Comparison of reduced Gröbner bases

Note that for the special case of tropical Grassmannians non-trivial points can also be computed by lifting point configurations via the tropical Stiefel map, see [HJS14, Proposition 12] and [FR15]. The reduced Gröbner basis of the initial ideal under w with respect to the reverse lexicographical ordering consists of 5543 binomials with degrees ranging from 2 to 7. The Gröbner cone $C_w(I)$ happens to be simplicial with 12 facets.

Figure 5 shows some data on the reduced Gröbner bases of the partially saturated initial ideals under weight vectors on the facets of $C_w(I)$, the rows represent binomials, trinomials and quadrimomials respectively and the columns represent degrees 2 to 7. The numbers in the center show the valency of the respective facet. The computation of the 12 tropical links using Algorithm 4.5 took 7 minutes, while all attempts to compute any of the 12 tropical prevarieties failed to terminate within several hours.

Using Algorithm 4.5, we conclude:

Theorem 4.8

The tropical Grassmannian $\mathcal{G}_{d,n}$ have links of valency bigger than three for $d = 3, 4$ and $n = 8$.

Proof. In the proof of Theorem 3.7, we have constructed two maximal cones which were not simplicial.

The maximal cone in $\mathcal{G}_{3,8}$ has 9 facets in total. Using Algorithm 4.5, we see that five have valency 4, two have valency 8 and there two links each with valency 16 and 24.

The maximal cone in $\mathcal{G}_{3,8}$ has 10 facets in total. Using Algorithm 4.5, we see that two have valency 4, two have valency 6, two have valency 16 and there are four links each with valency 8, 16, 20 and 46. \square

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